Mediterranean Journal for
Research in Mathematics Education
Vol. 5, 2 , 49-76, 2006

Primary School Teachers' beliefs and change of beliefs on Mathematical Infinity

This work is part of the wider research project: «*Methodological aspects (theoretical and empirical) of the in training and in service teachers working for every level of education*», financed by the University of Bologna.

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ABSTRACT***:*** *This study is included in the complex problems about the teaching and the learning of the concept of mathematical infinity pointing out the beliefs, the intuitive acceptances and the mental processes of primary school teachers. In particular the teachers' misconceptions can be considered like the starting of didactical obstacles that, with epistemological ones frequently pointed out in the international literature, create difficulties in the learning of this concept by secondary high school students. Moreover, such difficult didactical obstacles are hard to eliminate, unless one intervenes on primary school teachers' preparation in this specific field.*

1. INTRODUCTION

This paper is the synthesis of a research work carried out over several years on mathematical infinity. The first research results have been already published (Sbaragli, 2003, 2004). The present synthesis is mainly focussed on the beliefs and the corresponding changes of beliefs of primary school teachers on mathematical infinity. In particular, our aim is to point out that and to what extent this topic is unknown to teachers working for this school level both from the mathematical-epistemological point of view and from the cognitive one. Therefore, it is not uncommon to find, within the beliefs possessed by those teachers, numerous misconceptions[1] related to different fields of mathematics. Moreover, the probable change of conviction that some of those teachers experience, once they have been showed an elementary mathematical proof on mathematical infinity, clearly proves that infinity has never been considered as a learning subject or the topic for individual accurate reflection but it is exclusively grounded on spontaneous and intuitive beliefs only based on common sense. As a consequence, the result is an explicit manifestation that teachers show of a strong

[1] On the meaning of this word, see the contributions and critical considerations dealt with in D'Amore, Sbaragli (2005) and in Sbaragli (2005a).

personal cognitive disadvantage as regards this piece of knowledge, a disadvantage bringing about negative repercussions on the didactical transposition.

2. THEORETICAL FRAMEWORK

2.1 Beliefs and change of beliefs

The theoretical framework is to be considered within the research context of beliefs and change of beliefs in mathematics. Specific research studies in this field have sprung up quite recently, but they immediately revealed the great impact that this kind of reflections has on didactics.
Preliminarily, a necessary distinction, as it appears in D'Amore and Fandiño Pinilla (2004) and that we adopt, has to be made:
«- *belief*: an opinion, a set of judgements/expectations, what is thought about something;
- the set of beliefs of somebody (A) on something (T) gives the *conception* (K) of A relatively to T; if A belongs to a social group (S) and shares with the other members of S the set of beliefs relatively to T, then K is the conception of S relatively to T».
In the present paper we will be dealing only with those conceptions when T is the mathematics of infinity and we will take into account S as primary school teachers.
In this respect, it is important to recall the work of Schoenfeld (1992) who affirms that each individual conceptualises mathematics and approaches the mathematical world on the basis of her/his system of mathematical beliefs which implies that it is impossible to separate knowledge (on mathematics) from beliefs (on mathematics) in teachers' minds (see also: Fennema, Franke, 1992).

Among the other classic research studies in this field we mention: Thompson (1992) who offers a wide overview on the empirical researches on beliefs and beliefs; Hoyles (1992) who highlights that it is not possible to separate the analysis of the beliefs belonging to one individual from those shared by the social group she belongs to, since beliefs are nothing else than the result of complex interactions among social groups; Pehkonen (1994) illustrating a vast bibliography on this subject, particularly focussing on teachers' beliefs on mathematics and the corresponding change in the teaching methods; Da Ponte et al. (1999) providing an interesting and accurate overview on research carried out on mathematics teachers' beliefs and on the teaching-learning process of mathematics. Furthermore, crucial to the aim of our analysis is the study proposed by Leder et al. (2002) throughout twenty chapters all completely devoted to mathematical beliefs considered from different perspectives and the related synthesis of the research panorama in this field. In particular, the second section of this text is about teachers' beliefs. Other fundamental theoretical considerations on the structure of beliefs and on present research studies on this topic are those provided by Törner (2002).
Need mention here also the research studies of Llinares: that of 1996 where the author establishes a relation between beliefs and knowledge and that of 1999 focussed on the

dialectical relation among beliefs and teaching practice and revealing that it is difficult to identify a cause-effect relation i.e., if it is beliefs that influence the practice or if it goes the other way round. We quote here also the work dating 2002 where the author underlines the role of beliefs within the process of learning-how-to-teach according to the concept of a situated perspective: what is learnt depends on what is problematized and codified in the learning situations. In other words, the research reveals how beliefs (together with other factors) influence the learning process: what is learnt and how it is learnt.

The contributions of Tirosh and Graeber (2003) demonstrate that beliefs can create obstacles but can also serve as a powerful thrust towards change: a change in the personal teaching practice.

The nature of our research has involved us in investigating also the aspect of the "change of beliefs" in the sense of "development – evolution of beliefs over time" (Wilson, Cooney, 2002). On this respect, in Da Ponte et al. (1999) a synthesis of the specific research on the change of teachers' beliefs is included and also the change in the teaching methods of teachers, on which we will dedicate further investigation.

Moreover, this topic constituted the research subject of Strehle et al. (2002) dealing with the change of beliefs triggering the change of practice as a direct consequence of the introduction of information technologies in teaching practice and of Chapman (2002) presenting an interesting survey on teachers, turning out to be of great help for the theoretical basis of present work.

2.2 Mathematical Infinity: from epistemological to didactical obstacles

Numerous are the research studies in mathematics education, based on the issue of teaching and learning concepts related to mathematical infinity and considering as research subjects students in order to single out reasons making infinity such a difficult topic to be correctly constructed.

For the sake of brevity, we will not go over the wide literature available and we basically refer to D'Amore (1996, 1997). The articles in questions constituted the groundings of the work carried out at the ICME VIII, Seville 14-21 July 1996, considered highly representative of international research panorama at the time of reference, since the author was Chief Organizer of the Topic Group 14: *Infinite processes throughout the curriculum*. Thus, our starting point was an accurate outlook on different research "categories" and a vast bibliography with more than 300 titles that have been updated up to present days.

Great attention of the research has been devoted to students and many of the studies in this field before 1996 point out that from both the historical point of view and that of the learning of mathematical infinity, the evolution of the actual conception of infinity is extremely slow, frequently contradictory and only possible thanks to a process involving systematisation and cognitive maturation of human creations and learning processes concerning only a very limited number of individuals.

In this perspective, we basically refer to the following research works, all inspired by the classic philosophical debate on infinity in the potential and in the actual sense: Moreno and Waldegg (1991) investigating the "potential" and "actual" uses of the term "infinity", considered at times as an adjective and at times as a noun; Tsamir and Tirosh (1992) working on the difficulties encountered by students in the understanding of the idea of actual infinity; Shama and Movshovitz Hadar (1994) showing that when it is about counting, by the analysis of periodical phenomena you can come to a concept of infinity seen as a integer number; Bagni (1998 and 2001) pointing out the differences of the conception of potential and actual infinity and infinitesimal in students' conceptions before and after the study of Analysis; Garbin (2003) revealing university students' inconsistencies to deal with actual infinity.

All of these research works and many others, not mentioned in this article for the sake of synthesis, are aimed at the detection of epistemological obstacles due to students' primal intuitions which have turned out to be similar to those characterizing the history of mathematics and that constitute the key to understand some of the difficulties students have to face. On this respect, we recall the research work conducted by Schneider (1991) who analysed the epistemological obstacle of the heterogeneity of dimensions deriving by the infinite cuts of a surface or a solid with students aged 15-18. A further obstacle singled out by Duval (1983), always referring to infinity, is called *sliding* and it is about the difficulties students face to accept the biunivocal correspondence between N and its subset of even numbers. In this case the author talks about the phenomenon of sliding from the verb to Have to the verb to Be during proofs. In Arrigo and D'Amore (1999, 2002) the sliding effect is considered in a wider sense, that is to say, when during a demonstration you are talking about something (in a certain way or within the context of a specific language) and you suddenly turn/switch to another subject (in another way or using another language code).

Arrigo and D'Amore (1999, 2002) studies carried out with students attending the last years of the high secondary school revealed that their misconceptions are not merely due to epistemological obstacles but also to didactical ones. These two research works revealed a classic phenomenon already known as *dependence* according to which there are more points in a longer segment than in a shorter one. This misconception was already known to Fischbein, Tirosh and Hess since 1979 and further investigated in numerous later works (confront for example Fischbein, 1992a; 2001) and also analysed by Tall in a famous article of 1980. This phenomenon can be observed not only in geometrical field but it is also evident when referring to *dependence* of the cardinality on the "size" of numerical sets. For example, since the set of even numbers represents a sub-set of the natural numbers set, the former seems to be by implication formed of a smaller number of elements. Among the researches dealing with wrong intuitions and representations students build for themselves when trying to put into biunivocal correspondence infinite sets, we mention Tsamir and Tirosh (1994) and their successive article of 1997 focussing on the metacognitive aspect and consistency and Tall (2001). All these surveys highlight students' beliefs based on the truthfulness of the VIII Euclid's common notion: «The whole is greater than its parts», both for finite and infinity. This *dependence* phenomenon is based on the generalisation to infinite cases of what has been learnt of the biunivocal correspondence of finite cases (Shama and Movshovitz Hadar, 1994).

Another incorrect intuitive model revealed by Arrigo and D'Amore (1999, 2002), concerning high school students, is linked to the idea of a segment conceived as a *"necklace of beads"*, that is to say the segment is considered as a thread formed of beads-points put one next to the other. This model derives from the misconceptions of the mathematical point considered as an entity provided with dimension and from the wrong ideas related to the topology of the straight line [on this aspect we refer also to Tall's (1980) research work on students' beliefs regarding the number of points contained in a straight line and in a segment and Gimenez (1990) identifying primary school students' difficulty to conceive the density concept].

Another phenomenon treated in the works of Arrigo and D'Amore (1999, 2002) is known to the literature as *"flattening"*. This is about considering all infinite sets as having the same cardinality, that is to say that a biunivocal correspondence could be established between all infinite sets. This phenomenon has been already noted by Waldegg (1993) and Tsamir and Tirosh (1994) who concentrated on students' intuitions and representations when trying to establish a biunivocal correspondence between infinites sets. We also include the contributions provided by the School of Tel Aviv, with special reference to Efraim Fischbein and its followers (Fischbein, Jehiam, Cohen, 1994, 1995) who revealed students' difficulties to pass from the dense to the continuum, reporting the difficulty to properly construct the idea of irrational number. In more detail, literature on this subject has showed that once the students have accepted that two sets, such as N and Z for instance, have the same cardinality (thanks to the help of the researcher or teacher showing them the biunivocal correspondence between the two given sets), it is much more common that students tend to consider as true the generalisation that all infinite sets must have the same cardinality, which is not the case. The latter misconception is not only due to epistemological obstacles, but also to didactical obstacles as pointed out by Arrigo and D'Amore (1999, 2002).

Also the phenomenon of *flattening,* as well as that of *dependence,* is based on the generalisation to infinite case to what has been learnt about the biunivocal correspondence of finite cases (Shama and Movshovitz Hadar, 1994).

Our treatment should include also the study of Garbin (2005) on the incidence, not positive all the times, of representations and the different language codes of mathematics in the perception of the concept of infinity performed on students aged 16 - 20.

Interesting is the study of D'Amore et al. (2004) conducted in Colombia, Italy and Switzerland on the "sense of infinity", demonstrating that this sense does exist, but it can be reached only in some specific cases. Additionally, outcomes reveal that there is no link between the "sense of number" with the consequent skill to provide acceptable intuitive "estimates" and the "sense of infinity" with the consequent skill to provide intuitive "estimates" of infinite cardinalities.

Even more pertinent with this research work, is the research of Tsamir (2000) about teachers' misconceptions of infinity that can represent, for obvious reasons of knowledge transmission, didactical obstacles in students' learning. In particular, the author underlines that the difficulties to treat the concept of infinity in the actual sense rather than in the potential sense concern not only students but also in training teachers, and this magnifies the necessity to pay great consideration in future to the didactical obstacles and the learning contents to form the educational offer.

3. DESCRIPTION OF RESEARCH PROBLEMS

Here as follows we introduce the problems and corresponding questions inspiring this research study:

Q1. Are primary school teachers' beliefs on mathematical infinity those expected in the mathematical context or do they represent misconceptions diverging from the correct ones? More explicitly, do primary school teachers know the concept of infinity both in the mathematical – epistemological and in the cognitive sense? In particular, what do they exactly know about this topic?

Q2. In case teachers' misconceptions on infinity are detected, are they ascribable to every branch of mathematics (arithmetic and geometry in particular) or do they concern more specifically one of these fields? What kind of misconceptions are they?

Q3. Are some of the prospective misconceptions emerging among teachers, stable ones or are teachers open to change their mind, once it is granted to them the access to the correct mathematical knowledge, transferred by means of vocational courses or targeted interviews? As a consequence of this change of beliefs, are those teachers also available to change accordingly their teaching practice?

4. RESEARCH HYPOTHESES

Here as follows we report the hypotheses related to problems/questions described in paragraph 3:

H.1. Our viewpoint is that the majority of primary school teachers' beliefs are misconceptions related to the mathematical field. We believe that, on the basis of our workshops conducted with teachers, mathematical infinity is a rather unfamiliar subject to them, both from a mathematical - epistemological and from a cognitive point of view. We therefore think that teachers would not be able to handle infinity and to conceive it as a "mathematical object". Consequently, we assumed that teachers would stick to naive beliefs not possessing proper competence on this subject, beliefs turning out into real misconceptions.

H.2. Moreover, if the misconceptions assumed in H.1. are verified, we believe that these are related to every field of mathematics and in particular, to the same extent, to arithmetic and geometry. Our hypothesis is that teachers are stuck to naive beliefs such as *infinity* is simply *the indefinite, the unlimited* (belief deriving from a linguistic-semantic confusion between the adjectives "infinite", "unlimited" and "indefinite"), *a very large finite number* (whose consequence is to transfer the same modalities applicable to finite set also to infinite sets in fact considered as *very large* finite sets) or an exclusive *potential process.* We additionally assumed that we would be able to find in teachers' affirmations some misconceptions of *dependence* and *flattening,* besides "the necklace model", frequently indicated by students (even mature students) as a suitable model to mentally represent the points on the straight line and which have been at times reported by students (at the end of their secondary high school) as the model their primary school teachers provided them with, a model withstanding every further and successive attempts to rebuild it according to a more appropriate and correct knowledge (Arrigo and D'Amore, 1999; 2002).

H.3. We assumed that, if the two above-mentioned hypotheses had been proved true, a considerable number of teachers, once accessing a correct mathematical knowledge on infinity, would show their availability to change their beliefs, as indicator that this concept is merely based on naive intuitions that, though being deeply rooted are not supported by any kind of knowledge in this respect. In other words, we believe that teachers have no problem in admitting their cognitive gaps on this subject and therefore they are open to change their beliefs but only as a result of a targeted and correct education. In conclusion, teachers in questions are available to change their teaching activity as a result of an appropriate preparation and training in this field.

If the so far envisaged hypotheses had been proved true, we would have considered the possibility and the necessity of revising the didactical contents of primary school teachers' training courses. This is not merely meant to force teachers to change the cultural contents of their knowledge, but to prevent them from building intuitive models that could bring about situations of cognitive disadvantage for their students as revealed by the research.

5. Research Methodology. Teachers participating in the research, methodology and contents.

The survey involved 16 Italian teachers of primary schools [4 from Venice, 8 from Forlì, 4 from Bologna] adopting a methodology structured in various phases.

The first phase was based on the administration of a questionnaire to be filled individually formed of 13 A4 sheets, one sheet for each question (with space for teachers to write their answers). Questions were structured as follows: 11 were centred on the knowledge possessed by teachers on mathematical infinity; 2 on teachers' corresponding classroom didactical activity.

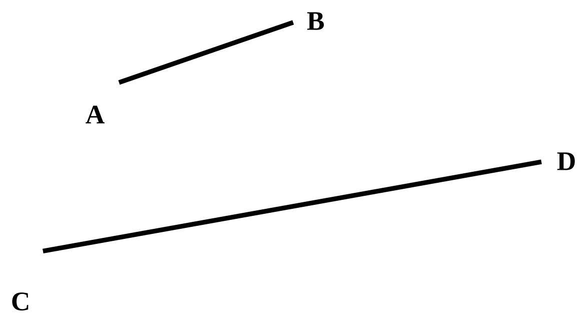
This article will describe only the 11 questions regarding teachers' competence on this topic whereas the other 2 concerning the classroom didactical transposition will be the object of a following article.

Here as follows we transcribe the 11 questions examined in this article:

1) *What do you think mathematical infinity means?*

2) *Does the concept of mathematical infinity relate both to the adjective infinite and to the noun*[2] *infinity?*

3) *Are there more points in the AB segment or in the CD segment? (Write down on the sheet of paper everything that comes to mind).*



4) *How many even numbers are there: 0, 2, 4, 6, 8, …?*

5) *How many odd numbers are there: 1, 3, 5, 7…?*

6) *How many natural numbers are there: 0, 1, 2, 3…?*

7) *How many multiples of 15 are there?*

8) *Are there more even or odd numbers?*

9) *Are there more even or natural numbers?*

10) *Are there more odd or natural numbers?*

11) *Are there more multiples of 15 or natural numbers?*

In consideration of the nature of this research and most of all the involved subjects' competences on this specific topic, the choice was not to establish a determined order of the questions to pose taking into account the preliminarity of concepts as the discrete that should precede the continuum. As a matter of fact, only question no. 3 seems to pertain explicitly and specifically to the field of "continuum infinity".

For the outline of the questionnaire, several informal interviews with teachers different from those taking part in the actual research study were held to verify the text readability and understandability. The teachers have judged the questionnaire easily "comprehensible". As a matter of fact, after a first reading of the questions, teachers unanimously affirmed that it was clear and of accessible interpretation, even though when it came to answer the very first question, 13 teachers, manifested great embarrassment: *«I don't know what to write, I never reasoned on this topic»*. Only after

[2] The term *infinito* is used in Italian both as a noun and as an adjective, thus covering both meanings of the English words *infinity* and *infinite.*

some self-assuring statements such as: *«I will write down what comes up to mind, even if it won't be well expressed»*, they started answering the first question.

Teachers had one hour for the questionnaire, so that teachers could read it through, reflect, think it over again and organise their answers with no pressure and taking their time. None of the teachers involved used all the time available.

Only after all sheets with answers had been handed in, the second phase based on discussion and confrontation started and with no time restrictions.

The questionnaire was therefore the starting point for the successive discussions on all the problems and aspects related to mathematical infinity which took place four times between the researcher and two teachers and twice between the researcher and four teachers (16 teachers in total). The tests have been administered in 6 different days. Discussion groups were organised as to allow confrontation between teachers that could already get on well together and were used to discuss and exchange opinions independently of the total number of teachers forming each group.

The decision of implementing confrontation between teachers, rather than between a single teacher and the researcher, is based on the necessity of collecting teachers' real beliefs on various topics, otherwise difficult to be identified. When teachers are asked to express or defend their own opinions, in front of other colleagues with whom they feel confident and are used to arguing and sharing more or less the same knowledge, the expected outcome is that they would feel freer to manifest their ideas. The applied strategy also served the aim of reducing some teachers' reactions such as: "trust in the researcher" or "trust in what mathematicians affirm" [often reported in literature; for example: Perret Clermont, Schubauer Leoni and Trognon (1992)], emerging not only when research is addressed to students but also when teachers are involved.

The discussion phase structured as described has proved to be far more fruitful than the written answers collected by means of the questionnaire. Such a complex and delicate topic needs a deep investigation into teachers' individual and single beliefs. To this aim, opinion exchange has proved to be a very useful means of revising and reworking the questionnaire's answers, to understand their intimate meaning, to verify their stability and to point out possible contradictions.

We adopted the technique of active open debate in groups of different size, using the recorder and leaving the researcher the task to direct the discussion, intervening only to stimulate debate on some relevant aspects pertaining to the issue. The intention was to discover teachers' beliefs, their doubts and perplexities, the deep-seated intuitive models, always bearing in mind in this part of the interview the researcher's task that was that of not changing teachers' viewpoint.

However, it was clearly stated right from the beginning that their names would not appear in the research work.

Successively, when the discussion was thoroughly exhausted with the explicitation of the beliefs of the teachers, the researcher assumed the role of educator, proposing learning situations, explaining concepts most of the times in contrast with previous teachers' beliefs.

In particular, two biunivocal correspondences between infinite sets and related to questions no. 3 and 9 of questionnaire were showed, these proofs were followed by two further questions (which we will thoroughly illustrate later on) which required individual answers. Specifically, the researcher showed the elementary version of the classic Georg Cantor's proof (1845-1918) related to question no. 3 that proves that there is the same number of points in two segments of different length (Courant, Robbins, 1941). In order to illustrate it, teachers were showed by the researcher-educator the biunivocal correspondence on a sheet of paper containing two segments AB and CD (differently positioned on the plane from those concerning question number no. 3; they were shifted by hypsometry so that they appeared parallel and "centred" with respect to one another). At the beginning, with the help of a ruler, the point O of intersection between the straight lines AC and BD was drawn; successively from O the points of the segment AB were projected on the segment CD and vice-versa. In so doing, the biunivocal correspondence between the sets of points belonging respectively to segments AB and CD was demonstrated. It was therefore easy to observe that there is the same number of points in segments of different length.

Successively teachers received a sheet of paper with the following question they were asked to answer individually:

12) *Try to be as honest as possible answering the following question: were you convinced by the proof that there are as many points in AB as in CD?*

After answer no. 12 had been handed in, teachers were shown by the researcher the proof of question no. 9, which proves that the set of even numbers (E) is formed by the same number of elements than that of natural numbers (N), showing the related biunivocal correspondence (Tall, 2001 pp. 213-216).

Let us illustrate the biunivocal correspondence showed to teachers:

N	0	1	2	3	4	5	…	n	…
	↕	↕	↕	↕	↕	↕	↕	↕	↕
E	0	2	4	6	8	10	…	2n	…

This idea developed from the consideration made by Galileo Galilei (1564-1642) (even if Galileo talked about square numbers and not even numbers): to each natural number corresponds a determined square and vice-versa to each square number corresponds a determined natural number (its «arithmetic root»). Thus, there are as many natural numbers as square numbers.

As in the case of the previous question, teachers received a sheet of paper with the following question they were asked to answer individually:

13) *Try to be as honest as possible answering the following question: were you convinced by the proof that there are as many numbers in the set of even as in that of natural numbers?*

Only after each teacher had handed in the answer to the last question, the researcher-educator, before inciting the discussion on the two illustrated proofs, introduced also the distinction between potential and actual infinity proposed by Aristotle. Only afterwards the discussion with teachers started in order to assess the verification of some possible changes in their beliefs as a consequence of this training programme centred on the three aspects above mentioned. In order to identify these changes, there was a prior evaluation of the answers given by teachers, the affirmations expressed with their colleagues and the statements expressed with the researcher.

6. Test results and opinion exchange

From the answers given to the questionnaire, some rather general affirmations emerged which have undergone further investigation thanks to the opinion exchange between teachers. Some of the answers given to each question, considered as the most significant ones, have been selected and are provided below. Integrations to such answers obtained through verbal exchange during discussion are provided as well. The aim is to offer the widest and most representative view as possible of the respondents' beliefs. Researcher's interventions and comments, which have been indicated with Res., served the aim of stimulating conversation and going more deeply into teachers' beliefs.

The following paragraph will be dealing with the teachers' change of beliefs emerged as the result of the provided training.

6.1 Description of questionnaire results and related opinion exchange

1) As to the answers to the first question of the questionnaire, we propose here as follows an exhaustive classification; this classification will show some aspects already known to international research but some others are new and therefore particularly interesting. Therefore, we will briefly discuss the answers ascribable to those categories already analysed by the research and we will particularly linger over the new aspects emerged. It ought to be noted that *none* of the 16 of the interviewed teachers had an appropriate and correct conception of infinity in the mathematical sense. It is important to remember that the operated classification should not be considered as definitive, since as we shall see later, some of the teachers' affirmations, that were at the beginning processed as belonging to a specific category, have been also successively inserted in other ones as a direct consequence of the outcome of successive conversations.

• ***Infinity as indefinite.*** 7 teachers tend to consider infinity as indefinite, that is to say they do not know how much it is, what it is exactly, what it represents.

R.: To me it means without boundaries, with no limits like the space.

Res.: In the sense of indefinite?

R.: Yes without borders.

C.: Something that you cannot say.

Res.: In what sense?

C.: You don't know how much it is.

A.: Something that cannot be written down.

• ***Infinity as a finite large number.*** 3 teachers affirm that infinity is nothing but a very large finite number.

A.: To me it's a large number, so large that you cannot say its exact value.

B.: After a while, when you are tired of counting you say infinity meaning an ever-increasing number.

• ***Infinity as unlimited.*** 5 teachers confuse infinity with unlimited, they think the term infinity can be attributed exclusively to the straight line, half-line and plane, i.e. everything unlimited. Therefore it is not possible to talk of infinity with regard to the set of points of a segment that is limited by its end-points. Curious enough is that if the researcher intervenes asking: *«How many points are there in a segment?»* teachers show to know the answer to the question that is: *«Infinite»*, but without understanding the real sense and mathematical meaning of this statement. As a matter of fact, further investigating it, 3 of the 5 respondents affirm that in the case of the number of points of a segment, infinity is considered a large finite number whose exact value is unknown, whereas the remaining 2 see infinity as indefinite: you do not know exactly how much it is. So, all of the 5 answers are pertinent also to the other categories indicated for this question. To these teachers the following relationship seemed to be valid: in the cases of lines, planes and space, "infinity" and "unlimited" seem to be synonyms; in the cases of the quantity of numbers or points, they refer to infinity as a very large finite number or indefinite number.

A.: With no limits.

M.: Something that I cannot quantitatively measure.

(Also in this case, teacher M. associates the term infinity with unlimited without considering that a segment, for instance, though being limited and measurable in the sense understood by M., contains infinite points).

N.: Something unlimited.

Res.: So you will never use the word infinity referring to a segment.

N.: No, because it has a beginning and an end.

Res.: How many points do you think there are in a segment?

N.: Oh you're right, infinite. But it's just to say a large number, not as large as in the straight line. Even if you make very little points, you cannot fit in it more than that.

Teacher N. already reveals the belief emerged also in answers to question no. 3, that is to say, that there are more points in a straight line than in a segment, stressing the idea that to greater length corresponds a greater number of points. Points are therefore not conceived as abstract entities but as objects that should have a certain dimension[3] in order to be represented.

G.: Unlimited.

Res.: How many points do you think there are in a segment?

G.: You say that in a segment there are infinite points because it is not known how many there are exactly.

• ***Infinity as process.*** Only a single teacher talks of infinity in the first question referring to a never-ending process:

B.: I know infinity, it means to keep going on as with numbers… for ever.

This belief recalls the idea of potential infinity that we shall see illustrated in next paragraph. Analysing the collected answers in more detail, it can be observed that also the answer given by B. or in other answers which we will be dealing with later on belong to the category of infinity as a large finite number. This belief is to be traced back to the concept of potential infinity considered as an ever-lasting process, this providing evidence that (as expected) beliefs of the same teacher can come under different categories.

2) The aim of the second question was to discover if awareness that infinity represents a mathematical object on its own existed among teachers.

For 13 teachers infinity in mathematics is only an adjective, the remaining 3 believe it is also a noun, but out of the latter 3, 2 of them conceive infinity as indefinite whereas the other one believes that you can use this word also as a noun but only meaning a very large finite number whose value is unknown.

N.: As an adjective.

M.: In mathematics it exists only as an adjective, in the Italian language also as a noun.

[3] This belief derives from the representation most commonly provided of the point (on the blackboard or in schoolbooks) conditioning the image students but also teachers possess of this mathematical object (Sbaragli, 2005b).

A.: In mathematics it is used as an adjective: infinite numbers, infinite space. As a noun in the Italian language: Leopardi's "Infinity",[4] *"I see infinity"; "I lose myself in infinity."*

B.: Also as a noun to mean a large number.

3) The third question was about the teachers' supposed belief that two segments of different length should correspond to a different number of points. This idea already emerged from the answer to the first question provided by the teacher N. and illustrated in point 1) of this paragraph.

All of the 16 interviewed teachers affirmed that in two segments of different length there is a different number of points and more specifically, to a greater length corresponds a greater number of points (as already investigated by Fischbein, 2001, p. 311). It stands out clearly that visually one segment appears to be included in the other and therefore in this case the figural model is predominant. This model negatively influences the answer and in fact the Euclidean notion: «The whole is greater than its parts» cannot be applied to infinity.

Here as follows, we have reported some of the answers pertaining to the above-mentioned belief:

N.: This makes me think that the different length of two segments should have some influence on the number of points.

B.: In the segment CD; of course, it's longer.

G.: In AB there should be a lot, in CD many more.

A.: I'm not sure. Given that the segment can be considered as a series of points in line, I think that CD has more points than AB, even if I have learnt that the point is a geometrical entity, which, being abstract, is not possible to quantify it because it's not measurable. I would say CD, anyway.

[The teacher A. showed inconsistency between what she affirms she had studied in order to take an Analysis exam at university and what she believes to be the most plausible answer according to common sense. Once again, the intuitive model persists and predominates. In this situation it is quite evident that there is no correspondence between the formal and the intuitive meaning (Fischbein, 1985, 1992b)].

The above discussed intuitive acceptance represents a widespread misconception, it has already been mentioned in paragraph 2 and is called *dependence* of transfinite cardinals on factors related to magnitudes (the set of greater size has more elements). The teacher in question is therefore convinced that a greater length implies also a greater cardinality of the set of points. This misconception, known to the research literature since a long time, will return also in the answers to questions no. 9-10-11, where *dependence* is to be intended as *dependence* of the cardinality on the "size" of numerical sets.

[4] Giacomo Leopardi's *Infinity* is one of the most famous Italian poems of all times.

Furthermore, these statements revealed a major presence of the so called "necklace model", already mentioned in this article, based on the idea of a segment conceived as a thread formed of very little beads-points, one very next to the other.

On this aspect, accurate researches have many times highlighted that mature students (those attending the last years of the high school or the first years of university) are not able to master the concept of continuity as a result of this persisting intuitive model provided by their teachers (Tall, 1980; Gimenez, 1990; Romero i Chesa and Azcárate Giménez, 1994; Arrigo and D'Amore, 1999, 2002). By collecting teachers provided answers, we could verify that this model not only constitutes a didactical strategic device preliminary to some more correct concept and used by teachers to provide their students with a certain idea of segment (though being them perfectly aware that this is only an approximate image completely distant from the real mathematical concept of segment) but it seems instead as the real model teachers possess of the segment and of the point and therefore they transfer this model as the definitive one. Conversations clearly revealed also serious deficiencies in teachers' competences mostly linked to the concepts of density and continuity of the ordered set of the points of the straight line.

Interesting on this respect is the following extract from a conversation:

B.: In the segment CD, of course it's longer (B. is talking of the number of points of two segments).

Res.: How many more?

B.: It depends on how big you make them.

M.: It depends on how you draw them: distant or very close to one another; but if you make them as close as possible and all of the same size then there are more in CD.

G.: In CD, it's longer.

Res.: But can you really see the points as graphically represented here?

G.: Yes, it's the kind of geometry we do that makes us see the points.

4) – 5) – 6) – 7) As to the four following questions, 15 teachers answered in this way: *«Infinite»*, but they actually ignore the sense of such an affirmation, with the exception of one of them who, after some hesitation, wrote: *«Quite a few!»*. The answer "Infinite" seems to derive from a common and most widespread attitude in mathematics, i.e. to answer with stereotyped answers distant from a real conceptual construction.

8) This question and the following questions no. 9) - 10) - 11) envisaged the task of comparing some infinite set cardinalities that are often dealt with in primary schools. The collected answers have been classified in the following three categories:

• ***There are as many even numbers as odd numbers.*** 12 teachers out of 16 have this opinion.

C.: It's the same number to me.

• ***It is impossible to make a comparison of the cardinalities of infinite sets.*** To 3 teachers a comparison of the cardinalities of infinite sets is not conceivable. As a matter of fact, in the logic of those who conceive infinity as indefinite or as something finite, very large but with an undetermined value, it is rather difficult if not impossible to make a comparison between cardinalities of infinite sets.

R.: You can not answer that, it is not possible to compare infinities.

• ***The unsure.*** One teacher answered back with a question:

A.: I would say they are of equal number, the even and the odd numbers; but I have a major doubt: If they are infinite how can I quantify them?

(From this answer emerges the idea of infinity seen as indefinite).

9) – 10) – 11) The answers provided to these three questions belong to the following four categories. It has to be underlined that all of the 16 interviewed teachers are consistent in always replying in the same way to all three questions:

• ***There are more natural numbers.*** 10 teachers answered that there are more natural numbers, supporting the common Euclidean notion: «The whole is greater than its parts».

C.: The natural numbers.

• ***You cannot compare infinite sets.*** The same 3, who in reply to question no. 8 could not conceive a comparison between cardinalities of infinite sets, remained firm in this opinion; this results in the idea that you can refer to cardinality only when dealing with finite:

R.: You cannot answer that, you can't make a comparison.

• ***The unsure.*** The same teacher, who answered to question no. 8 with another question, replied in the same way which shows consistency:

A.: I would say the natural numbers, but how can I quantify them? To say infinity means nothing.

• ***They are all infinite sets.*** 2 teachers affirmed that all the sets in question are infinite and therefore they have all the same cardinality.

B.: They are both infinite. If two sets are infinite, they're just infinite and that's it.

From the interview of these two teachers emerges the misconception of the *flattening* of transfinite cardinals illustrated in paragraph 2, resulting in the belief of considering all infinite sets of equal power. In other words, these teachers came spontaneously to the conclusion that being all the above-mentioned sets infinite, the attribute "greater", in compliance with a passage of Galileo's, cannot be used when dealing with infinities. The direct consequence is that all of the sets of this type are nothing else than infinite.

Res.: According to you, do all infinite sets have the same cardinality?

B.: What do you mean? The same number? Yes, if they are infinite!

6.2 The change of belief

In order to assess the real level of belief relatively to the affirmations expressed by teachers on mathematical infinity, two proofs have been illustrated referred to questions no. 3 and 9 followed by the distribution of questions 12 and 13, reported in paragraph 5, together with the explanation of the distinction between potential and actual infinity. The aim of this knowledge transmission was to evaluate if the interviewed teachers were open to change their mind with respect to their beliefs, as a clear indicator that their knowledge was kind of weak an unstable, or if on the contrary they were still convinced about what they have initially affirmed.

Answers to the questions posed after the above mentioned training and the handing in of the first 11 questions included in the questionnaire, integrated with the affirmations of the final discussion, are reported as follows.

12) After having showed the construction described in paragraph 5 corresponding to the question no. 3, demonstrating that there is the same number of points in two segments of different length, the question no. 12 has been distributed.
The answers collected by this question, which have been successively integrated with the final discussion, belong to the following categories:

• ***Not convinced by the proof.*** 5 out of 16 respondents were not convinced by the proof:
Res.: Were you convinced by this proof?

M.: Well, not really; to me a point is a point, even if I make it smaller, it's still a point. Look! (Drawing a black spot on the sheet). *Then, if I make them all of the same size, how can they be of the same number?*

Res.: According to you, between two points is there always another one?

M.: No if the two points are immediately next to one another. If I draw two points one next to the other, very close, so close, practically stuck to one another there won't be any in between them.

B.: Ummh! But in the segment AB you go over the same point when lines get thicker. I'm not convinced.

As a matter of fact, to grasp the exact meaning of this construction has proved quite a difficult task for those teachers to whom the point is not conceived as an abstract entity with no dimension, but rather as the mark left by the pencil and therefore with its own dimension (Tsamir, 1997; Acuña, 2005). More in general terms, teachers rejecting the above discussed construction are those who imagine the segment as the "model of the necklace of beads".

• ***Convinced by proof.*** 9 were convinced by the proof. The teachers A. and C., in particular, considered it crystal clear and extremely effective:

A.: That's nice! You convinced me.

C.: You convinced me, it's exactly like that.

Although these 9 teachers were promptly and immediately confident with the proof's correctness some doubts and perplexities were provoked by questions such as: «*Are you really sure about it?*». Our intention was to observe if teachers were inclined to change once again their mind showing by that not a profound and stable belief but only "trust" in the affirmation of the researcher. As a matter of fact, 3 admitted not being thoroughly convinced at this point, returning to the initial affirmation that there are more points in CD. [On this aspect of the belief change and its relative stability consult: Arrigo and D'Amore (1999, 2002)].

Res.: Are you really sure about it?

G.: No, no! I'm still convinced that there are more of them in CD, you can see it.

R.: I'm not so sure.

These constant changes of belief provide evidence that teachers' knowledge on mathematical infinity is not firm and stable.

• ***Trust in mathematicians.*** One teacher showed a sort of "trust in mathematicians", though not being totally convinced by the proof:

A.: If you mathematicians say that, we trust you!

• ***The unsure.*** One teacher seemed to be in need of some kind of explanation, but after a little discussion claimed to be convinced:

M.: It's because you took that point over there, if you had taken another one it wouldn't have worked out… look!

The teacher drew another spot-point different from the projection point identified by the researcher and then drew half-lines intersecting the longer segment and not the shorter one. These considerations mirror the difficulty in understanding what it is and how a mathematical proof works but this aspect goes outside the scope of our analysis.

Res.: Yes, but if you want the projection point to be exactly the point you drew you have to perform a translation of the two segments and project right from that point (the translation on M.'s drawing was performed)*, however, the translation will not alter the number of points of the two segments.*

M.: Ok, you convinced me.

13) The biunivocal correspondence was subsequently demonstrated to the 16 teachers proving that the cardinality of even numbers is the same as that of natural numbers and then the question 13 was asked.

Teachers reacted in two different ways which have been successively integrated with the final discussion:

• ***The dubious.*** 6 expressed themselves as not being particularly convinced:

M.: Well, it's kind of a "strain".

N.: It's strange, in the set of even numbers all the odd numbers are missing to obtain the natural ones.

• ***Those affirming to be convinced.*** 10 claimed to be convinced, but 2 in particular showed some "trust in the researcher" as the one who possesses knowledge.

Furthermore, during interviews, it emerged that all the teachers who accepted the idea that some infinite sets are of equal power (as in the case of the even and natural numbers) are now convinced that this is always valid for infinity and as a consequence they generalise that all infinite sets are also of equal number. This *flattening* misconception is seen as an "improvement" in comparison to the *dependence* misconception of the cardinality on the set "size". This change in attitude seems a slow and gradual approach towards the correct model of infinity. The appearance of the *flattening* misconception of transfinite cardinals was not unexpected since primary school teachers ignore the set of real numbers, and therefore they opt for a generalisation of the notions related to the sets known to them. They have actually no possibility to access the knowledge concerning the difference between the numerable cardinality and the continuum cardinality.

Prove of that is given by the following conversation:

A.: Therefore all infinite sets are equal.

Res.: What do you mean? Do integer numbers have the same cardinality as natural numbers?

A.: Uhm, yes.

Res.: And the rationals? The fractions.

A.: I think so.

Res.: And real numbers? The numbers that you obtain using the roots?

A.: Yes all, all of them, they are either all equal, that is to say infinite, or none of them is so.

These reflections reveal that in teachers' intuitive acceptance coexist some inconsistencies such as, *flattening* and *dependence* that are clearly in contrast with one another. It ought to be noted that there is a generalised difficulty encountered by teachers to actually realize the contradiction emerging by two statements which we attribute to the lack of knowledge and mastery of the concept of mathematical infinity. As a matter of fact, the non controllability of inconsistent situations when dealing with the concept of infinity has been already investigated by the international research (D'Amore and Martini, 1998; Tall, 1990; Tirosh, 1990; Tsamir and Tirosh, 1994, 1997, 1999; Garbin, 2003).

Potential and actual infinity. The opinion exchange revealed how some of the teachers' beliefs are definitely referable to the potential view of infinity. As a matter of fact, also in those cases when they adopted definitions ascribable to actual infinity such as: *«The straight line is formed of infinite points»*, they successively turned out to be inconsistent when declaring also that the term straight line is used only to indicate an ever longer segment, returning once again to the potential vision.

R.: We use to say that natural numbers are infinite, but we know that this doesn't mean a thing as they can't be quantified! It's like saying a very large number that you cannot even say; that you can go on forever I mean. To say straight line is like saying nothing, it doesn't really exist; it's another way of saying an ever longer line.

The choice was to let teachers grasp the double nature of infinity: potential and actual, to assess their possible change of beliefs on this respect. This stimulus provoked the following reactions:

• 10 teachers were still stuck to potential infinity being, according to them, nearer to the sensible world as shown in the following passage:

M.: To me there exists only the potential infinity, the other doesn't exist, it's pure fantasy, tell me, where is it?

Res.: When talking of the straight line ...

M.: But where is the straight line? There is none. So actual infinity does not exist.

Res.: What do you think of the straight line?

M.: I think these kinds of things shouldn't be taught, at least not in primary school, poor children what can they do! Yes, of course you can also say that the straight line is formed of infinite points, but how are they supposed to understand that? (I don't believe it myself!), at their age they have to see things. They have to touch things with their own hands.

N.: I think very large things though still finite exist, all the rest does not exist.

• 6 teachers seemed to grasp the idea of actual infinity. In particular, three teachers showed a very enthusiastic reaction to their discovery of the distinction between the two conceptions of infinity.

A.: I never thought about this distinction, but now I got it, I can imagine it.

B.: I never even thought about it, nobody gave me the possibility of reflecting on this topic, but to be honest I always thought that it was meant only in the sense of a continuous and constant process. But now I've understood the difference.

The latter statement shows the embarrassment felt by the teachers who were not given any possibility of reflecting on such fundamental topics they should be able to master in order to prevent the creation of students' misconceptions.

Many of the teachers involved in this research have clearly voiced their disadvantage with respect to this topic and the corresponding wish and need to know more about it. In this respect, we report two teachers' opinions:

M.: We need someone to help us reflect on such things and on the importance of transposing them in a correct way. In the mathematics we learned, they did not make us think about these things. We need some basic theory.

A.: Our problem is that we try to simplify things, without some previous theory. We are sure we've got it, but in fact we don't have it. We are concerned with transferring it in a tangible way, without deeply investigating how it works. But how can we do if nobody helps us changing our teaching method?

7. ANSWERS TO QUESTIONS FORMULATED IN PARAGRAPH 3

We are finally able to provide answers to the research questions formulated in 3.

A1. Primary school teachers possess various misconceptions on mathematical infinity. There is a total absence of knowledge of what is intended by mathematical infinity, both in the mathematical - epistemological and cognitive meaning as a consequence of a total lack of specific preparation in this field. To primary school teachers, infinity is an unknown concept, solely managed by common sense and for this reason considered as a banal extension of finite. That causes the creation of intuitive models that turn out to be thorough misconceptions. No specific mathematical knowledge possessed by teachers on this issue has emerged. As a matter of fact, there is a total absence of specific courses targeted to in training primary school teachers dealing with the fundamental importance of this issue in the teaching – learning process.

A2. Primary school teachers possess misconceptions of mathematical infinity in every fields of mathematics, but namely and in particular in the arithmetical and geometrical field with the same incidence. Teachers accept namely the Euclidean notion: «The whole is greater than its parts» for the finite and tend to consider it also valid for infinity, which is a *dependence* misconception concerning both fields. Expressions such as "to be a proper subset" and "to have less elements" should not be confused when dealing with infinite sets. Nevertheless primary school teachers, during their educational career, have only found evidence of what happens when dealing with finite and accepted it as an absolute intuitive model. In other words teachers tend to apply to infinity every concept valid for finite: if an A set is a proper subset of a B set, then the cardinality of B is automatically greater than the cardinality of A. In building such a misconception, teachers' intuitive model of the segment seen as a necklace of beads also plays a role, thus leading to the phenomenon of *dependence* on magnitudes. The *flattening* misconception also participates to a great extent in this mechanism, but in the didactical repercussion it brings about less affecting consequences than *dependence* to primary school pupils. Also the straight line seen as an unlimited figure and the never-ending counting of natural numbers seem to make teachers consider infinity only in power and not in act, which results in major didactical obstacles for future students. To this list we include also teachers' beliefs on mathematical infinity considered as a synonym of unlimited, indeterminate and very large finite number.

A3. A considerable number of teachers who revealed to possess misconceptions demonstrated, after an initial training, to be willing to change their mind i.e. to modify their beliefs, frankly admitting their deficiencies as regards this topic and actually voicing their need to know more about these concepts. These teachers declared themselves to be available to change their teaching practice if supported by a targeted training on the issues related to infinity. This aspect will be the object of a next article. In order to avoid such obstacles, in our opinion and also in the opinion of teachers themselves a better training is needed, so that a purely and exclusively intuitive approach to infinity can be averted. It is therefore necessary to reconsider the teaching contents for in training teachers working for any educational level. In so doing, it could be avoided that students in higher secondary school would have to face the study of Analysis with an improper background of misconceptions. The treatment of problems concerning actual infinity requires the development of different intuitive models, if not even opposite to those regarding finite.

8. CONCLUSIONS

The present research revealed specific misconceptions possessed by primary school teachers on mathematical infinity revealing a total lack of knowledge about this specific subject. Many teachers involved in this research project openly admitted to ignore this topic and they voiced their urgent need to know more about the issue in question.

The crucial point is that "no sensible magnitude is infinite" and therefore the comprehension of such topics seems to go against intuition and everyday experience (Gilbert and Rouche, 2001) and are also unfamiliar to who has never reflected on such topics.

Tsamir (2000) states: «Cantor's set theory and the concept of actual infinity are considered as opposite to intuition and can raise perplexities. Therefore they are not easy to be acquired and some special didactical sensitivity is necessary to teach them». In other words, if primary school teachers have never been taught the topic in question, they are obviously bound to refer, when teaching these concepts, to their intuitions and common sense that the history of mathematics has vastly proved to be opposite to theory. (Mathematical infinity received a definite systematization from a scientific viewpoint only at the end of the XIX century, therefore quite recently in comparison to its long history lasting over two millennia). Consequently, Tsamir's didactical sensitivity would hardly be developed in primary school teachers without a proper and targeted preparation.

The main consequence of such conceptions in the teaching activity is the risk of providing students with images completely extraneous to mathematics and possibly turning into obstacles to future learning both in analysis courses of higher school and even before in the lower secondary when concepts such as the density of Q, irrational numbers, the ratio between the square side and its diagonal, the continuity of R or of the set of the points of the straight line and many more are introduced. As a matter of fact, when entering high school, students would literally "clash" into the necessary actual conception of infinity that may represent (and in fact it actually represents) an insurmountable conceptual difficulty, as the direct consequence of the possible construction in previous years of an intuitive model of infinity so deeply rooted and solely based on their own intuitions and those of their teachers, beliefs that are alas very distant from the world of mathematics.

We believe that the difficulties encountered by students, as revealed by numerous research works, in the understanding of the concept of mathematical infinity are not exclusively due to epistemological obstacles, but didactical obstacles resulting from the teachers' intuitive ideas magnify them also. It is also very likely that surveyed deficiencies on this specific topic are not a problem exclusively affecting primary school, but are instead rather widespread at every school level, among all those teachers who have never been given the opportunity to properly reflect on mathematical infinity.

So far, it seems as if such a topic had been very much underestimated, above all as a subject for teachers' training. Models provoking obstacles in the teachers' as well as students' minds are necessarily to be inhibited and overcome. Therefore, teachers' targeted training courses are required based on the peculiarities of infinity as well as on the outcomes of the research in Mathematics Education in this field. These courses should take into account the several intuitive aspects and peculiarities of infinity as well as the outcomes collected by the researchers of Mathematics Education. They should be mainly based on open and free discussion; the historical aspects of the subject should be outlined too. They should start from initial intuitive ideas in order to transform them into new and fully-fledged beliefs.

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ACKNOWLEDGMENTS: I want to express my greatest and heartfelt gratitude to Professor Bruno D'Amore, inspirer of present work and patient reader of this article. I want to thank him for all the changes and integrations he suggested me and for his contribution as to the bibliographical reference, by helping me selecting some of the texts that I have included in this bibliography.